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NOTE ON ASYMPTOTIC EXPANSIONS

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NOTE ON ASYMPTOTIC EXPANSIONS

Jürgen Moser

§1. We consider a system of linear differential equations which depends on a parameter ϵ.

$$(1) \qquad \frac{dy}{dt} = A(t,\epsilon)y = \epsilon^{-p}\{A_0(t) + \epsilon A_1(t) + \ldots\}y$$

when $y = (y_1, \ldots, y_n)$ is an n-vector and A_ν $n \times n$ matrices. If the parameter ϵ is considered small it is well known that the solution of such a system can be described by asymptotic – in general divergent – series expansions. The expansions contain except for powers of ϵ with rational exponents exponential terms of the form

$$e^{\chi(t,\epsilon)}$$

where $\chi(t,\epsilon)$ is a polynomial in $\epsilon^{\frac{1}{q}}$, q being a positive integer. This theory has a vast literature for which we refer to the paper of H. L. Turritin [1]. While in many expositions of this subject strong assumptions on the eigenvalues of $A_0(t)$ (e.g. to be distinct and different from zero) limit the generality considerably, it is the aim of Turritin's paper to give a theory in full generality (excluding turning points). This leads, on the other hand, to many involved matrix calculations and the reduction is rather complicated. The main difficulty seems to stem from nilpotent matrices $A_0(t)$.

In this note we study a question which is related to the presence of nilpotent matrices $A_0(t)$ and which aims at the description of the principal part of ~~the exponentials~~ $\chi(t,\epsilon)$ (~~i.e.~~ the terms of highest order in ϵ^{-1}). It is known that in case the eigenvalues of $A_0(t)$ are distinct and different from zero that

$$\frac{d\chi}{dt} = \lambda_\nu(t)\epsilon^{-p} + \ldots$$

where the λ_ν are the eigenvalues of $A_0(t)$. It is surprising that λ_ν , which are invariants under similarity transformations, play a role in the asymptotic expansion and, therefore, should be invariant under coordinate transformations

(2) $$y = T(t, \epsilon)z \ .$$

Such a transformation maps (1) into

$$\frac{dz}{dt} = B(t, \epsilon)z$$

where

(3) $$B = T^{-1}AT - T^{-1}\dot{T} \ .$$

This equation (3) defines an equivalence between A and B which differs from a similarity transformation. It is our aim to characterize the principal part of $\dot{\chi}(t, \epsilon)$ invariantly under (2) .

In (2) we admit power series $\epsilon^{\frac{1}{q}}$ $(q = 1 \ , \ 2 \ , \ . \ . \ )$ so that

$$B = \epsilon^{-m} \{B_0 + \ldots\}$$

is a series in fractional powers of ϵ . The rational number m will be called the <u>order</u> of B . It is our aim to minimize the order m under transformations (2) and we define

$$\mu = \underset{B \sim A}{\mathrm{Min}} \ m \quad \text{(if positive, otherwise } \mu = 0)$$

as the minimal values of m . That this minimum is attained will be shown by a construction which allows its computation.

A matrix B is called minimal if $m = \mu$. It will be shown that for

$\mu > 0$ B is minimal if, and only if

$$f(\lambda, x, \epsilon) = \det(\lambda I - B_0) \neq \lambda^n$$

and that this polynomial $f(\lambda, x, \epsilon)$ is invariant under the equivalence (3) This result can be interpreted in the following way: The equivalence (3) deviates from a similarity and, in fact, the additional term $T^{-1}\dot{T}$ might introduce terms of higher order than were present in A . If, however, A is minimal, then these terms are necessarily nilpotent. Conversely, if the highest order terms in A are nilpotent, then they can be removed and p diminished.

We discuss a trivial example: The matrix

$$A(t, \epsilon) = \epsilon^{-p}\begin{pmatrix} t & -1 \\ t^2 & -t \end{pmatrix}$$

is nilpotent, i.e. all eigenvalues zero. This does not imply that the asymptotic expansion is free from exponential terms (i.e. $\chi \equiv 0$) and, in fact, one finds a solution

$$y_1 = \exp(\pm\epsilon^{-\frac{p}{2}}t), \quad y_2 = (t\epsilon^{-\frac{p}{2}} \mp 1)\exp(\pm\epsilon^{-\frac{p}{2}}t).$$

This shows that $\mu = \frac{p}{2}$ which can be easily obtained by the method to be discussed.

The following considerations are algebraic in nature and, therefore, all series can be considered convergent. This result can be used in a general theory of asymptotic expansions which will not be done here, since a treatment of this kind can be found in Hukuhara's paper [2]* .

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* I am grateful to Professor Sibuya for this reference.

§2 Necessary and sufficient conditions for minimal B .

To simplify the following considerations, we assume all elements of the matrices $A_\nu(t)$ to be meromorphic functions of t * . The complex variable ranges over a fixed domain D .

We denote by F_0^q the space of functions

$$f(t,\epsilon) = Q(t)^{-1} \sum_{\nu=0}^{\infty} f_\nu(t)\epsilon^{\frac{\nu}{q}}$$

where f_ν , $Q(t)$ are analytic in D and the series converges in $|\epsilon|<\epsilon_0$, $t \in D$. These functions form a ring and quotients of such functions will have the form

$$f = Q^{-1} \sum_{\nu=-\nu_0}^{\infty} f_\nu(t)\epsilon^{\frac{\nu}{q}}$$

which forms the space F^q . Similarly, the matrices T , whose elements lie in F_0^q , F^q , define the spaces M_0^q , M^q respectively. The union of all M^q for $q = 1, 2, \ldots$ will be denoted by M .

The important property of F^q is that addition, subtraction, division and differentiation does not lead out of F^q .

A matrix $T \in M_0^q$, i.e.

$$T = \sum_{\nu=0}^{\infty} \eta^\nu T_\nu \ , \quad \eta = \epsilon^{\frac{1}{q}}$$

is called a <u>unit</u> if T^{-1} also lies in M_0^q . This is the case if, and only if,

$$\det T_0(t) \neq 0 \quad \text{in } D \ .$$

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* Under this assumption it is not necessary to shrink the t-domain which is usually done in order to avoid that eigenvalues cross each other.

Lemma 1: If $\det T \neq 0$ is an element of M^q, then it can be written in the form

(4) $$T = P(t, \epsilon)\, \epsilon^{\alpha}\, Q(t, \epsilon)$$

where P, Q are units in M_0^q (i.e. $\det P_0 \neq 0$, $\det Q_0 \neq 0$) and

$$\epsilon^{\alpha} = \operatorname{diag}(\epsilon^{\alpha_1}, \epsilon^{\alpha_2}, \ldots, \epsilon^{\alpha_n}) \quad (\alpha_1 \leq \alpha_2 \leq \ldots \leq \alpha_n)$$

and $q\alpha_\nu$ are integers.

Proof: Without loss of generality we can assume $q = 1$; otherwise let $\eta = \epsilon^{\frac{1}{q}}$.

The matrix Q will be built up from "elementary matrices" consisting of permutation matrices (independent of t, ϵ) whose determinant is ± 1 and matrices which differ from the unit matrix in one off diagonal element. All these matrices as well as their product are units, since their determinant is ± 1.

Now determine α_1 as the greatest integer, such that $\epsilon^{-\alpha_1} T$ is a power series in ϵ, i.e. $\epsilon^{-\alpha_1} T \in M^1_0$. Thus the constant term in $\epsilon^{-\alpha_1} T$ is non-zero. Applying a permutation matrix Q, we can achieve that

$$TQ_1 = \epsilon^{\alpha_1}(s_1, s_2, \ldots, s_n)$$

where $s_\nu = s_\nu(t, \epsilon)$ are column vectors containing no negative powers of ϵ and $s_1(t, 0) \neq 0$.

Now choose α_2 as the maximal integer such that

$$TQ = (\epsilon^{\alpha_1} r, \epsilon^{\alpha_2} r_2, \epsilon^{\alpha_2} r_3, \ldots, \epsilon^{\alpha} r_n)$$

where Q is any product of elementary matrices and r_ν are column vectors. Obviously, $\alpha_2 > \alpha_1$. After having defined $\alpha_1, \ldots, \alpha_{\ell-1}$ we maximize a_ℓ under all products of elementary matrices Q such that

$$(5) \qquad TQ = (\epsilon^{a_1} p_1, \ldots\ldots, \epsilon^{a_{\ell-1}} p_{\ell-1}, \epsilon^{a_\ell} p_\ell, \epsilon^{a_\ell} p_{\ell+1}, \ldots, \epsilon^{a_\ell} p_n)$$

and the elements of these column vectors belong to $F_0{}^1$.

The existence of such a maximal a_ℓ follows from the assumption

$$\det T = c(t)\epsilon^\gamma + \ldots \qquad c(t) \not\equiv 0 \ .$$

γ gives an upper bound for a_ℓ since

$$a_1 + a_2 + \ldots + a_{\ell-1} + (n - \ell + 1)\, a_\ell \leq \gamma \ .$$

One proves by induction that the rank of the matrix

$$(p_1(t,0), \ldots, p_n(t,0))$$

over the meromorphic functions is at least ℓ . Otherwise one can achieve by an elementary matrix that $p_{\nu*}(\cdot, 0) \equiv 0$ for some $\nu* < \ell$ which means that a_ℓ can be enlarged. Thus for $\ell = n$ one has

$$TQ = (p_1, \ldots, p_n)\ \epsilon^a = P(t,\epsilon)\ \epsilon^a$$

$$\det P(t,0) \neq 0 \ .$$

Observe that with Q also Q^{-1} is an elementary matrix, hence a unit. This proves the Lemma.

Remark: The above representation is not unique but it is easily seen that the a_ν are unique.

Theorem 1: Let

$$(6) \qquad \begin{cases} A = \epsilon^{-p}\, \{A_0(t) + \ldots\} \\ B = \epsilon^{-p}\, \{B_0(t) + \ldots\} \end{cases}$$

be two matrices in M which are equivalent, i.e. related by (3) with a $T \in M$, p is assumed to be a positive rational number.

Then

$$\det(\lambda I - A_0(t)) = \det(\lambda I - B_0(t))$$

Proof: Assume A contains integer powers of $\epsilon^{\frac{1}{q_1}}$ and B in $\epsilon^{\frac{1}{q_2}}$ then define η by

$$\epsilon = \eta^{q_1 q_2}$$

so that A , B contain only integer powers of η . Let T be the transformation relating A and B by (3) . Represent T in the form (5)

$$T = P\eta^{\alpha}Q$$

and let

$$\hat{A} = P^{-1}AP - P^{-1}\dot{P}$$

$$\hat{B} = QBQ^{-1} + \dot{Q}Q^{-1}$$

then $\hat{A}$ is equivalent to A and $\hat{B}$ equivalent to B . Moreover, since P , Q are units, P^{-1} , Q^{-1} are power series and so are the terms $P^{-1}\dot{P}$ and $\dot{Q}Q^{-1}$. Since $p > 0$ it follows that

$$\hat{A} = \epsilon^{-p}\{P_0^{-1}A_0P_0 + \ldots\}$$

$$\hat{B} = \epsilon^{-p}\{Q_0B_0Q_0^{-1} + \ldots\}$$

and

$$(7) \qquad \begin{cases} \det(\lambda I - \hat{A}_0) = \det(\lambda I - A_0) \\ \det(\lambda I - \hat{B}_0) = \det(\lambda I - B_0) \end{cases} .$$

Finally, $\hat{A}$, $\hat{B}$ are related by the similarity transformation

$$\hat{B} = \eta^{-\alpha}\hat{A}\eta^{\alpha}$$

from which it follows that the first determinant in (7)

$$\det(\lambda I - \hat{A})\big|_{\epsilon=0} = \det(\lambda I - \hat{A}_0)$$

agrees with the second, which proves the Theorem 1 .

Definition: A matrix

$$A = \epsilon^{-p}\{A_0 + \ldots\} \in M \; ; \; p > 0$$

is called minimal, if for each $T \in M$

$$B = T^{-1}AT - T^{-1}\dot{T} = \epsilon^{-m}\{B_0 + \ldots\}$$

one has

$$m \geq p > 0 \; .$$

Otherwise B is called not minimal, for $p > 0$. For $p > 0$ A is defined to be minimal unconditionally.

Consequence of Theorem 1: If A is not minimal, then A_0 is nilpotent.

Proof: Since A is not minimal, there is a $T \in M$ such that

$$B = T^{-1}AT - T^{-1}\dot{T}$$

does not contain terms of order $\geq m$ in ϵ^{-1}, i.e. in (6) one has $B_0 \equiv 0$. Hence by the Theorem 1

$$\det(\lambda I - A_0) = \det(\lambda I - 0) = \lambda^n \, ,$$

which expresses that A_0 is nilpotent.

Theorem 2: Every matrix

$$A = \epsilon^{-p}\{A_0(t) \ldots\} \in M$$

is equivalent to a minimal matrix B .

Proof: This theorem requires a construction and guarantees the existence of a minimal value μ of m . We make use of a procedure which has been used by Hukuhara [2] in a similar connection. The use of Theorem 1 will simplify Hukuhara's Lemma.

It is well known that each component of y satisfies a single differential equation of order $\leq n$, to construct such a differential equation for, say, y_1, we set

$$z_1 = y_1$$

$$z_2 = \frac{dy_1}{dt} = \sum_{\nu=1}^{n} a_{1\nu}(t,\epsilon) y_\nu$$

and express y_1, y_2 in terms of z_1, z_2 if $a_{12} \neq 0$. If $a_{1\nu^*} \neq 0$ for some $\nu^* > 1$ renumber $y_2, \ldots, y_n$, to reduce this to the previous case. If $a_{1\nu} \equiv 0$ for $\nu = 2, \ldots, n$, $z_2 = \frac{dy_1}{dt} = a_1 \; y_1$ which represents the desired differential equation. Otherwise let

$$z_3 = \frac{dz_2}{dt} = \sum \dot{a}_{1\nu} y_\nu + \sum a_{1\nu} \dot{y}_\nu$$

which is a linear combination of $y_1, y_2, \ldots, y_n$ and hence a linear combination of $z_1, z_2, y_3, \ldots, y_n$. If the coefficients of $y_3, \ldots, y_n$ are not all zero, we can assume by renumbering that the coefficient of y_3 is not identically zero, which allows to express y_1, y_2, y_3 in terms of $z_1, z_2, z_3, z_4, \ldots, y_n$.

This process of elimination can be continued until $\frac{dz_\tau}{dt}$ is a linear combination of $z_1, \ldots, z_\tau$. If $\tau = n$, one obtains for z a system

$$\dot{z} = Bz$$

where

$$B = \begin{pmatrix} 0 & 1 & 0 & & 0 \\ 0 & 0 & 1 & & \\ & & & 1 & \\ & & & & \\ & & & & 1 \\ b_\tau & & & & b_1 \end{pmatrix} \tag{8}$$

If, however, $\tau < n$ one can continue the same procedure by defining

$$z_{\tau+1} = y_{\tau+1} ,$$

introducing

$$z_{\tau+2} = \frac{dz_{\tau+1}}{dt} .$$

and eliminating the further y_ν for $\nu > \tau$. This leads to a system

$$\dot{z} = B(t, \epsilon) z$$

where

$$(9) \qquad B = \begin{pmatrix} B_{11} & 0 & 0 \\ B_{21} & B_{22} & \\ & & 0 \\ B_{\kappa 1} & & B_{\kappa\kappa} \end{pmatrix}$$

Here the $B_{\nu\nu}$ ($\nu = 1, \ldots \kappa$) are square matrices of the type (8) while the $B_{\nu\mu}$ ($\nu > \mu$) contain nonzero elements only in the last column.

The variables y and z are related by linear equations, the coefficients of which are obtained from those A by addition, multiplication, division and differentiation, hence belong to F^q if $A \in M^q$. Since the relation is invertible, we have

$$y = Tz$$

with

$$(10) \qquad T \in M^q ; \ \det T \neq 0$$

i.e. A and B are equivalent.

It is easily shown that B can be transformed into a minimal matrix by a diagonal matrix of the form

$$E = (\epsilon^{\gamma_1} , \ldots , \epsilon^{\gamma_n}) .$$

First, one can easily achieve that $B_{\nu\mu}$ for $\nu > \mu$ does not contain negative powers of ϵ by a transformation

(11) $$E = \mathrm{diag}\,(E_{11}\ ,\ E_{22}\ ,\ \ldots\ ,\ E_{\kappa\kappa})$$

where the $E_{\nu\nu}$ are square matrices (corresponding to the block representation (9) of B) and

$$E_{\nu\nu} = I\,\epsilon^{-\beta_\nu}\ .$$

The matrix $E^{-1}\,B\,E$ is obtained from B by replacing $B_{\nu\mu}$ by $B_{\nu\mu}\,\epsilon^{\beta_\nu-\beta_\mu}$. Choosing for β_ν an appropriate increasing sequence, one can achieve that $B_{\nu\mu} \in M_0^q$.

If in (9) no $B_{\nu\nu}$ contains negative powers of ϵ , then $B \in M_0^q$ and B is minimal by definition. Otherwise, if $B_{\nu\nu}$, which is of the form (8) , contains negative powers of ϵ , determine the smallest rational number γ_ν such that

$$b_1\,\epsilon^{\gamma_\nu}\ ,\ b_2\,\epsilon^{2\gamma_\nu}\ ,\ \ldots\ ,\ b_\tau\,\epsilon^{\tau\gamma_\nu}$$

contain no negative powers of ϵ . Thus at least one of these terms does not vanish for $\epsilon = 0$. By definition, $\gamma_\nu > 0$ is a rational number. Let

$$\gamma = \operatorname*{Max}_\nu \gamma_\nu$$

and apply the transformation with

$$E_{\nu\nu} = \mathrm{diag}\,(1\ ,\ \epsilon^{-\gamma}\ ,\ \ldots\ ,\ \epsilon^{-\gamma(\tau-1)})\ .$$

Since

$$E_{\nu\nu}^{-1} B_{\nu\nu} E_{\nu\nu} = \epsilon^{-\gamma} \begin{vmatrix} 0 & 1 & & & \\ & & 1 & & \\ & & & 1 & \\ & & & & 1 \\ c_\tau & & & & c_1 \end{vmatrix}$$

with

$$c_\mu = b_\mu e^{\mu\gamma}$$

and at least one of them does not vanish at $\epsilon = 0$. Since the characteristic polynomial of $\epsilon^\gamma \cdot E_{\nu\nu}^{-1} B_{\nu\nu} E_{\nu\nu}$ is $\lambda^\tau - c_1, \lambda^{\tau-1} - \ldots - c_\tau$ it is not identically λ^τ for $\epsilon = 0$ and for some ν. Thus we obtain

$$E^{-1} B E = \epsilon^{-\gamma} \{C_0 + \ldots\}$$

when

$$\det (\lambda I - C_0) \not\equiv \lambda^n$$

which proves that $E^{-1} B E$ is minimal. From theorem 1, it follows that $\gamma > p$. This proves theorem 2.

<u>Definition:</u> For a given matrix A let

$$B = \epsilon^{-m} \{B_0 + \ldots\}$$

be an equivalent minimal one. Then define

$$\mu = \mu(A) = \begin{cases} m & \text{if } m \geq 0 \\ 0 & \text{if } m < 0 \end{cases}.$$

The number μ has by definition an invariant meaning. In fact, it is the minimal order of matrices which are equivalent to A. As a consequence of the proof of theorem 2, we show

<u>Theorem 3:</u> For a given (not necessarily minimal) matrix $A \in \mathcal{M}^q$ one can compute $\mu = \mu(A)$ by transforming it into the form (9) by a transformation $T \in M^q$ and determining the smallest $m \geq 0$ such that

$$\det(\lambda I - \epsilon^m B) \in M_0 \quad .$$

Then
$$m = \mu \quad .$$

<u>Proof:</u> Since

$$C = E^{-1} B E = \epsilon^{-\mu}(C_0 + \dots)$$

is minimal it follows that

$$\lambda^n \equiv \det(\lambda I - C_0) = \det(\lambda I - \epsilon^{\mu} B)\Big|_{\epsilon = 0}$$

which implies $m = \mu$.

Theorem 1 implies that the characteristic polynomial of a minimal matrix is independent of the representation if $\mu > 0$, which proves the invariance of the nonzero eigenvalues of a minimal matrix under equivalence.

It is easily verified that these nonzero eigenvalues agree with $\epsilon^{\mu} \frac{d\chi}{dt}\Big|_{\epsilon = 0}$ where χ was defined in the introduction. If $\mu = 0$ the eigenvalues do not play any role, in fact, in that case A is equivalent to $B \equiv 0$. To prove this statement let

$$\frac{dy}{dt} = A y$$

where $A \in M_0^q$ does not contain negative powers of ϵ . It is known that in this case the matrix solution of

$$\frac{d}{dt} Y = A Y \, ; \, Y = I \text{ for } \epsilon = 0$$

belongs to M_0^q , and is a unit, since $\det Y_0 = 1$.

If one makes the transformation

$$T = Y$$

$$y = Tz = Yz$$

every solution y goes over into a vector z which is independent of t, namely

$$Ay = \dot{y} = \dot{Y}z + Y\dot{z} = AYz + Y\dot{z}$$

hence

$$\dot{z} = 0$$

which proves the statement

§3 We mention without proof that one can obtain the principal part of those exponentials χ which start with terms of lower order than $\epsilon^{-\mu}$ by refining the above method. Instead of minimizing the order of A as defined above, one can minimize a "matrix order" or order which is defined as follows: Assuming for simplicity $\det A \neq 0$ we can represent A by the previous Lemma as

$$P \, \epsilon^{-m} \, Q$$

where $m = \operatorname{diag}(m_1, m_2, \ldots, m_n)$ $m_1 \geq m_2 \geq \ldots \geq m_n$. Ordering m lexicographically, i.e.

$$m > \hat{m} \quad \text{if} \quad m_\nu = \hat{m}_\nu \quad \nu < \kappa$$
$$\text{and} \quad m_\kappa > \hat{m}_\kappa$$

one can find conditions on minimal matrix m which lead to a description the principal parts of the other exponentials.

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